

Smooth actions of Lie groups and Lie algebras on manifolds

Morris W. Hirsch
 Mathematics Department
 University of Wisconsin at Madison
 University of California at Berkeley

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*Dedicated to Professor Richard Palais with warmest regards
 on the occasion of his eightieth birthday.*

Abstract

Necessary or sufficient conditions are presented for the existence of various types of actions of Lie groups and Lie algebras on manifolds.

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1 Introduction

Lie algebras were introduced by Sophus Lie under the name “infinitesimal group,” meaning a finite dimensional transitive Lie algebra of analytic vector fields on an open set in \mathbb{R}^n . In his 1880 paper *Theorie der Transformationsgruppen* [36, 22] and the later book with F. Engel [37], Lie classified infinitesimal groups acting in dimensions one and two up to analytic coordinate changes. This work stimulated much research, but attention soon shifted to the structure, classification and representation of abstract Lie algebras and Lie groups.

There are relatively few papers on nonlinear actions by noncompact Lie groups (other than \mathbb{R} and \mathbb{C}). A selection is included in the References.

We avoid the important but difficult classification problems, looking instead for connections between algebraic invariants of Lie algebras, topological invariants of manifolds, and dynamical properties of actions. The motivating questions are whether a given Lie group or Lie algebra acts effectively on a given manifold, how smooth such actions can be, and what can be said about orbits and kernels.

Background

In 1950 Mostow [44] completed Lie’s program of classifying effective transitive surface actions. One of his major results is:

Theorem 1.1 (Mostow) *A surface M without boundary admits a transitive Lie group action iff¹ M is a plane, sphere, cylinder, torus, projective plane, Möbius strip or Klein bottle.²*

By a curious coincidence these are the only surfaces without boundary admitting effective actions of $SO(2)$, according to a well known folk theorem.³

We mention a far-reaching extension of Theorem 1.1 that deserves to be better known:

Theorem 1.2 *Let G be a Lie group and H a closed subgroup such that G/H is compact. Then $\chi(G/H) \geq 0$, and if $\chi(G/H) > 0$ then the fundamental group of G/H is finite.*

This is due to Gorbatsevich *et al.* [12] (Part II, Chap. 5, p. 174, Cor. 1). See also Hermann [22], Felix *et al.* [18, Prop. 32.10], Halperin [19], Mostow [45], Samelson [49].

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¹We use the late Professor Halmos’ symbol “iff” for “if and only if.”

²For each equivalence class of transitive surface actions, Mostow gives a representative basis of vector fields. Determining which of these classes contains a specified Lie algebra can be nontrivial. Here the succinct summary in M. Belliard [4] is helpful.

³The key points in the proof are that the action is isometric for any metric obtained by averaging a Riemannian metric, and the existence of arcs transverse to nonconstant orbits (Whitney [62]).

Terminology

The sets of integers, positive integers and natural numbers are denoted by \mathbb{Z} , $\mathbb{N}_+ = \{1, 2, \dots\}$ and $\mathbb{N} = \mathbb{N}_+ \cup 0$ respectively. i, j, k, l, m, n, r denote natural numbers, assumed positive unless the contrary is indicated. \mathbb{F} stands for the real field \mathbb{R} or the complex field \mathbb{C} . Vector spaces and Lie algebras are real and finite dimensional, and manifolds and Lie groups are connected, unless otherwise noted. The kernel of a homomorphism h is denoted by $\ker(h)$.

A topological manifold is a locally Euclidean metric space. Unless otherwise noted, manifolds are assumed to be analytic. M^n denotes a real or complex analytic manifold having dimension n over the ground field \mathbb{R} or \mathbb{C} . The boundary of M is ∂M . Except as otherwise indicated, manifolds are connected and maps between manifolds are C^∞ . The tangent vector space to M at $p \in M$ is $T_p(M)$. A vector field on M is always assumed to be tangent to ∂M .

“Group” and “algebra” are shorthand for “Lie group” and “Lie algebra”. G denotes a Lie group with Lie algebra \mathfrak{g} and universal covering group \widetilde{G} . Groups are assumed connected unless the contrary is indicated. The subscript “o” denotes the identity component. Lie groups are named by capital Roman letters and their Lie algebras are named by the corresponding lowercase gothic letters.

$GL(m, \mathbb{F})$ is the group of $m \times m$ invertible matrices over \mathbb{F} ; its Lie algebra is $\mathfrak{gl}(m, \mathbb{F})$. The subgroup of unimodular matrices is $SL(n, \mathbb{F})$, and that of the unimodular upper triangular matrices is $ST(m, \mathbb{F})$. The k -fold direct product $G \times \cdots \times G$ is denoted by G^k . The universal covering group of G is \widetilde{G} .

The commutator subgroup of G is G' . The upper central series is recursively defined by $G^{(0)} = G$, $G^{(j+1)} = (G^{(j)})'$, with corresponding Lie algebras $\mathfrak{g}^{(j)}$. Recall that G and \mathfrak{g} are *solvable*, of *derived length* $l = \ell(\mathfrak{g}) = \ell(G)$, if $l \in \mathbb{N}_+$ is the smallest number satisfying $\mathfrak{g}^{(l)} = 0$. For example, $\ell(\mathfrak{st}(m, \mathbb{F})) = m + 1$.

G and \mathfrak{g} are *nilpotent* if there exists $k \in \mathbb{N}$ such that $\mathfrak{g}_{(k)} = \{0\}$, where $\mathfrak{g}_{(0)} = \mathfrak{g}$ and $\mathfrak{g}_{(j+1)} = [\mathfrak{g}, \mathfrak{g}_{(j)}]$. It is known that \mathfrak{g} is solvable if and only if \mathfrak{g}' is nilpotent (Jacobson [33, Corollary II.7.2]).

Actions and local actions An *action* α of G on M , denoted by (G, M, α) , is a homomorphism $g \mapsto g^\alpha$ from G to the group of homeomorphisms of M , having a continuous *evaluation map*

$$\text{ev}_\alpha: G \times M \rightarrow M, (g, x) \mapsto g^\alpha(x).$$

The action is called C^s when ev_α is differentiable of class C^s , where $s \in \mathbb{N}$, $s = \infty$, or $s = \omega$ (meaning analytic). “Smooth” is a synonym for C^∞ . A *flow* is an action of \mathbb{R} .

If $1 \leq r \leq \omega$, a C^r *Lie algebra action* β of \mathfrak{g} on M , recorded as (\mathfrak{g}, M, β) , is a linear map $X \mapsto X^\beta$ from \mathfrak{g} to $\mathfrak{v}^r(M)$ that commutes with Lie brackets and whose evaluation map is C^r . Unless otherwise indicated it is tacitly assumed that $r = \infty$ or ω . The action is *complete* provided each vector field X^β is complete, i.e., all its integral curves extend over \mathbb{R} .

An *n-action* (of a Lie group or Lie algebra) is an action on an n -dimensional manifold.

A C^s *local action* λ of G on V , ($0 \leq s \leq \omega$) is a homomorphism $g \mapsto g^\lambda$ from G to the groupoid of C^s diffeomorphisms between open subsets, having the following properties: The evaluation map $(g, x) \mapsto g^\lambda(x)$ defines a C^s map $\Omega \rightarrow V$, where Ω is an open neighborhood of $\{e\} \times V$. Suppose $s > 1$. Corresponding to λ is a C^{s-1} action of \mathfrak{g} on V denoted by $d\lambda$. Conversely, every C^s action of \mathfrak{g} comes from a C^s local actions of G . When G is simply connected and the Lie algebra action (\mathfrak{g}, M, β) is a complete, then there exists an action (G, M, α) such that $\beta = d\alpha$. For results on the smoothness of these actions see Hart [20, 21], Stowe [53].

The *orbit* of $p \in M$ under (G, M, α) is $\{g^\alpha(p) : g \in G\}$, and the orbit of p under a Lie algebra action (\mathfrak{g}, M, β) is the union over $X \in \mathfrak{g}$ of the integral curves of p for X^β . An action is *transitive* if it has only one orbit.

The *fixed point set* of (G, M, α) is the set

$$\text{Fix}(\alpha) := \{x \in M : g^\alpha(x) = x, (g \in G)\},$$

denoted also by $\text{Fix}(G^\alpha)$. For Lie algebra actions $(\mathfrak{g}, M, \alpha)$ the fixed point set is

$$\text{Fix}(\beta) := \text{Fix}(\mathfrak{g}^\beta) := \{p \in M : X_p^\beta = 0, (X \in \mathfrak{g})\}$$

Thus $p \in \text{Fix}(\beta)$ iff p is a fixed point for the local flows on M defined by the vector fields X^β for all $X \in \mathfrak{g}$.

The *support* of an action γ on M is the closure of $M \setminus \text{Fix}(\gamma)$.

An action α is *effective* if $\ker(\alpha)$ is trivial, and *nondegenerate* if the fixed point set of every nontrivial element has empty interior. A degenerate action α of \mathfrak{g} is trivial if α is analytic or \mathfrak{g} is simple.

A group action is *almost effective* if its kernel is discrete.

2 Constructions of actions

Analytic actions of \mathbb{R}^n

It is true, but not easy to prove, that every real analytic manifold admits a nontrivial analytic vector field.⁴ In fact the following holds:

Theorem 2.1 *The vector group \mathbb{R}^n has an effective analytic action on every real analytic manifold M of dimension ≥ 2 .*

The proof relies on the theory of approximation of smooth functions by analytic functions (Grauert [17]).

Lemma 2.2 *Let $f : M \rightarrow \mathbb{R}$ be a nonconstant analytic function that is constant on each boundary component. Then there exists a nontrivial analytic vector field X on M such that $df(X_p) = 0$ for all $p \in M$, and X generates an analytic flow that preserves each level set of f .*

*Proof.*⁵ First consider the case that M is an open set W in a half-space $H^d = [0, \infty) \times \mathbb{R}^{d-1} \subset \mathbb{R}^d$, so that $\partial W = W \cap (\{0\} \times \mathbb{R}^{d-1})$. Fix a nonconstant analytic function $f : W \rightarrow \mathbb{R}$ that is constant on each boundary component. The analytic vector field $Y = \frac{\partial f}{\partial x_2} \frac{\partial}{\partial x_1} - \frac{\partial f}{\partial x_1} \frac{\partial}{\partial x_2}$ is nontrivial, annihilates df , and is tangent to ∂W . Endow W with a complete Riemannian metric and denote the norm of $\xi \in TW$ by $|\xi|$. There is a nonconstant analytic function $u : W \rightarrow \mathbb{R}$ such that $\sup_{q \in M} |u(q)Y_q| < \infty$. The vector field $X = uY$ is complete, and satisfies the Lemma.

Now let M be arbitrary. By Whitney's embedding theorem and the tubular neighborhood theorem (see Hirsch [26]) we take M to be an analytic submanifold of some halfspace $H^d \subset \mathbb{R}^d$ such that

⁴This is false for complex manifolds, e.g., Riemann surfaces of genus > 1 .

⁵Joint work with Professor Joel Robbin.

$M \cap \partial H^d = \partial M$, with an open neighborhood $W \subset H^d$ of M having an analytic retraction $\pi: W \rightarrow M$ taking ∂M into ∂H^d . By the first part of the proof, there is a complete, nontrivial analytic vector field U on W that annihilates the function $f \circ \pi: W \rightarrow \mathbb{R}$. For $p \in M$ set $U_p = X_p + Z_p$ with $X_p, Z_p \in T_p M$ and $d\pi_p Z_p = 0$. The maps $p \mapsto U_p$ are analytic vector fields on M . We have

$$\begin{aligned} 0 &= d(f \circ \pi)_p U_p = df_p \circ d\pi_p X_p = df_p \circ d(\pi|_M)_p X_p, \\ &= df_p X_p \end{aligned}$$

because $\pi|_M$ is the identity map. ■

Proof of Theorem 2.1. Choose an analytic vector field X on M as in the Lemma. Fix analytic functions $u_j: M \rightarrow \mathbb{R}$, $j = 1, \dots, n$ that are linearly independent over \mathbb{R} such that $|u_j X|$ is bounded. The vector fields L_j on M defined by $L_j(p) = u_j(f(p))X_p$ are complete and therefore generate flows ϕ_j . In each level set V of f , $L_j|_V$ is a constant scalar multiple of $X|_V$. Therefore ϕ_j preserves V , and $[L_j, L_k] = 0$. This shows that the ϕ_j generate an analytic action Φ of the group \mathbb{R}^n .

To show that Φ is effective, assume $a_j \in \mathbb{R}$ are such that $\sum_j a_j L_j$ vanishes identically, which means $(\sum_j a_j u_j(f(p)))X_p$ vanishes identically. So therefore does $\sum_j a_j u_j(f(p))$, by analyticity, because $X_p \neq 0$ in a dense open set. It follows that the a_j are zero because the u_j are linearly independent. ■

Lie algebra actions on noncompact manifolds

A manifold is *open* if it is connected, noncompact and without boundary. On many open manifolds it is comparatively easy to produce Lie algebra actions that are effective and analytic:

Theorem 2.3 *An open manifold M^n admits an effective Lie algebra action $(\mathfrak{g}, M^n, \beta)$ if there is an effective action $(\mathfrak{g}, W^n, \alpha)$ such that one of following conditions is satisfied:*

- (a) M^n is parallelizable (which holds if $n = 2$ and M^n is orientable)
- (b) $n = 2$ and W^2 is nonorientable.

In each case β can be chosen to be nondegenerate, analytic, transitive or fixed-point free provided α has the same property.

Proof We will define β as the pullback of α by an analytic immersion $M^n \rightarrow W^n$. The fundamental theorem of immersion theory (Hirsch [24, 25], Poenaru [48], Adachi [1]) says that such an immersion exists provided M^n is an open manifold, and the tangent bundle TM^n is isomorphic to the pullback of TW^n by map $f: M^n \rightarrow W^n$. In case (a) take f to be any constant map. For case (b) we first show that M^2 immerses in the Möbius band B^2 . To see this, note that every open surface has the homotopy type of a 1-dimensional simplicial complex, whence the classification of vector bundles implies TM^2 is the pullback of TP^2 by a map f from M^2 to the projective plane P^2 . As M^2 is an open surface, it can be deformed into an arbitrary neighborhood of its 1-skeleton, hence f can be chosen to miss a point of P^2 and thus have its image in a Möbius band. This shows that M^2 immerses in B^2 , and the conclusion follows because B^2 immerses in every nonorientable surface. ■

The real form of a Lie algebra \mathfrak{g} of matrices over \mathbb{C} or the quaternions \mathbb{H} is denoted by $\mathfrak{g}_{\mathbb{R}}$. From the natural projective actions of matrix groups we obtain:

Corollary 2.4 *The following effective kinds of analytic actions exist:*

- (a) $\mathfrak{sl}(3, \mathbb{R})$ and $\mathfrak{sl}(2, \mathbb{C})_{\mathbb{R}}$ on all open surfaces, and on all open parallelizable k -manifolds for $k \geq 3$,
- (b) $\mathfrak{sl}(n, \mathbb{R})$ on all open parallelizable k -manifolds, $k \geq n - 1$,
- (c) $\mathfrak{sl}(n, \mathbb{C})_{\mathbb{R}}$ on all open parallelizable k -manifolds, $k \geq 2n - 2$,
- (d) $\mathfrak{sl}(n, \mathbb{H})_{\mathbb{R}}$ on all open parallelizable k -manifolds, $k \geq 4n - 4$.

3 Lie contractible groups

Let \mathcal{G} denote either a Lie group G or its Lie algebra \mathfrak{g} . A *deformation* of \mathcal{G} is a 1-parameter family $\theta = \{\theta_t\}_{t \in \mathbb{R}}$ of endomorphisms $\theta_t: \mathcal{G} \rightarrow \mathcal{G}$ having the following properties:

- (D1) θ_t is the identity automorphism if $t \leq 0$,
- (D2) $\theta_t = \theta_1$ if $t \geq 1$.
- (D3) the map $\mathbb{R} \times \mathcal{G} \rightarrow \mathcal{G}$, $(t, g) \mapsto \theta_t(g)$ is C^∞ .

If θ is a deformation of G , the family $\theta'_t(e)\}_{t \in \mathbb{R}}$ of derivatives at the unit element $e \in G$ constitute a deformation θ' of \mathfrak{g} . When G is simply connected every deformation of \mathfrak{g} comes in this way from a unique deformation of G .

An *Lie contraction* of \mathcal{G} is a deformation θ such that θ_1 is the trivial endomorphism, in which case \mathcal{G} is called *Lie contractible*. It can be shown that this implies \mathcal{G} is solvable, and G is contractible as a topological space. It is easy to see that the direct product of finitely many Lie contractible groups is Lie contractible.

We prove below that $\mathfrak{st}(n, \mathbb{F})$ and its commutator ideal, which is nilpotent, are Lie contractible. But K. DeKimpe [8] pointed out that some nilpotent Lie algebras have unipotent derivation algebras, ruling out Lie contractibility. Goodman [14] cites an example due to Müller-Römer [46] of such an algebra, namely, the 7-dimensional Lie algebra with a basis such that

$$\begin{aligned} [X_1, X_k] &= X_{k+1}, \quad (k = 2, \dots, 6), \\ [X_2, X_3] &= X_6, \quad [X_2, X_4] = X_7, \\ [X_3, X_4] &= X_7, \quad [X_2, X_5] = -X_7. \end{aligned}$$

See also Ancochea & Campoamor [2], Dixmier & Lister [9], Dyer [10].

Let $\mathcal{H} \subset \mathcal{G}$ be a subalgebra or subgroup. A deformation θ of \mathcal{G} is a *retraction* of \mathcal{G} into \mathcal{H} provided

$$\theta_1(\mathcal{G}) \subset \mathcal{H}, \quad \theta_t(\mathcal{H}) \subset \mathcal{H} \text{ for all } t \in \mathbb{R}.$$

When such a θ is given and ψ is a deformation of \mathcal{H} , there is deformation $\psi \# \theta$ of \mathcal{G} , and the *concatenation* of ψ and θ , defined by

$$(\psi \# \theta)_t = \begin{cases} \theta_{2t} & \text{if } -\infty < t \leq 1/2, \\ \psi_{2t-1} \circ \theta_1 & \text{if } 1/2 \leq t < \infty \end{cases}$$

is also another retraction of \mathcal{G} into \mathcal{H} . Note that $(\psi\#\theta)_1 = \psi_1 \circ \theta_1$. If θ is a Lie contraction of \mathcal{H} then $\psi\#\theta$ is a Lie contraction of \mathcal{G} .

Theorem 3.1 *The groups $\widetilde{ST}_\circ(N, \mathbb{F})$, $ST'(n, \mathbb{F})$ and their Lie algebras are Lie contractible.*

Proof It suffices to prove the Lie algebras are Lie contractible. Give $\mathfrak{t}(n, \mathbb{F})$ the basis $\{T^{(ij)}\}_{1 \leq i \leq j \leq n}$ where the unique nonzero entry of the $n \times n$ matrix $T^{(ij)}$ is 1 in row i , column j . The matrices $T^{(ij)}$ with $i < j$ form a basis for the commutator ideal $\mathfrak{st}(n, \mathbb{F})'$, and $\{T^{(11)}, \dots, T^{(n-1, n-1)}\}$ is a basis for the subalgebra $\mathfrak{d}(n, \mathbb{R}) \subset \mathfrak{s}(n, \mathbb{R})$ of diagonal matrices.

For $1 \leq i < j \leq n$ and $t \in \mathbb{R}$, the functions

$$c_{ij}: \mathbb{R} \rightarrow [0, 1], \quad c_{ij}(t) := \begin{cases} 1 & \text{if } t \leq 0, \\ \exp \frac{(j-i)t}{1-t} & \text{if } 0 < t < 1, \\ 0 & \text{if } t \geq 1 \end{cases} \quad (1)$$

are C^∞ , analytic in the open interval $]0, 1[$, and flat at all $t \notin]0, 1[$. They satisfy

$$c_{ij}(t) \cdot c_{jk}(t) = c_{ik}(t) \quad (i \leq j \leq k). \quad (2)$$

Consider the 1-parameter family of linear maps

$$\theta_t: \mathfrak{st}(n, \mathbb{F}) \rightarrow \mathfrak{st}(n, \mathbb{F}), \quad (t \in \mathbb{R}),$$

defined on the basis elements by

$$\theta_t(T^{(ij)}) = \begin{cases} T^{(ij)} & \text{if } t \leq 0 \text{ or } i = j, \\ c^{(ij)}(t) \cdot T^{(ij)} & \text{if } 0 < t < 1 \text{ and } i < j, \\ 0 & \text{if } t \geq 1 \text{ and } i < j. \end{cases} \quad (3)$$

The first equation in (3) implies θ maps $\mathfrak{st}(n, \mathbb{F})'$ into itself and reduces to the identity deformation of $\mathfrak{d}(n, \mathbb{R})$.

Equation (3) defines a retraction θ of $\mathfrak{st}(n, \mathbb{F})$ into $\mathfrak{d}(n, \mathbb{F})$, thanks to (2), and θ restricts to a Lie contraction θ^1 of $\mathfrak{st}(n, \mathbb{F})'$. To obtain a Lie contraction of $\mathfrak{st}(n, \mathbb{F})$ it suffices to form a concatenation $\psi\#\theta$ where ψ is an algebraic contraction of $\mathfrak{d}(n, \mathbb{F})$. For example set

$$\psi_t(T^{(ii)}) = c(t) \cdot T^{(ii)}, \quad (i = 1, \dots, n)$$

where $c(t) := c_{21}(t)$ from Equation (1). ■

Deformations of actions

Let α_0, α_1 be actions of G on M . A *deformation* of α_0 to α_1 is a 1-parameter family of actions $\beta = \{(G, M, \beta_t)\}_{t \in \mathbb{R}}$ such that

- $\beta_t = \alpha_0, \quad (t \leq 0)$

- $\beta_t = \alpha_1$, ($t \geq 1$)
- the map $\mathbb{R} \times G \times M \rightarrow M$, $(t, g, x) \mapsto g^{\beta_t}(x)$, is C^∞ .

A Lie contraction θ of G determines the deformation β of α_0 to the trivial action, defined by

$$\beta_t = \alpha_0 \circ \theta_t,$$

indicated by

Theorem 3.2 *Let G be a Lie contractible group having an almost effective smooth action on \mathbf{S}^{n-1} . Then on every topological n -manifold M^n there is an effective action of G which is the identity outside a coordinate ball, and which is smooth if M^n is smooth.*

Proof Let θ be a Lie contraction of an effective smooth action $(G, \mathbf{S}^{n-1}, \alpha)$. An action $(G, \mathbf{S}^{n-1} \times \mathbb{R}, \beta)$ is defined by

$$g^\beta: (x, t) = (\theta_t(g)^\alpha(x), t).$$

β is smooth and effective, and

$$g^\beta: (x, t) = \begin{cases} (g^\alpha(x), t) & \text{if } t \leq 0, \\ (x, t) & \text{if } t \geq 1. \end{cases}$$

Transfer β to an action $(G, \mathbb{R}^n \setminus \{0\}, \gamma_0)$ by the diffeomorphism

$$\mathbb{R}^n \setminus \{0\} \rightarrow \mathbf{S}^{n-1} \times \mathbb{R}, \quad (x, t) \mapsto e^{-t}x.$$

This action extends to a smooth effective action $(G, \mathbb{R}^n, \gamma)$ which is the identity outside the unit ball. It can therefore be transferred to the desired action on M^n . ■

Corollary 3.3 *For all $n \geq 2$, $k \geq 1$ there are effective smooth actions of $ST_o(n, \mathbb{R})^k$ on every smooth n -manifold and of $\widetilde{ST}(n, \mathbb{C})$ on every smooth $2n$ -manifold.*

Proof $ST_o(n, \mathbb{R})$ and $\widetilde{ST}(n, \mathbb{C})$ are Lie contractible (Theorem 3.1) and have effective smooth actions on \mathbf{S}^{n-1} and \mathbf{S}^{2n-1} respectively. By Theorem 3.2 there are k coordinate balls with disjoint closures in M^n (respectively, M^{2n}) that support effective smooth actions of $ST_o(n, \mathbb{R})$ (respectively, $\widetilde{ST}(n, \mathbb{C})$). The desired actions are obtained by letting j 'th factor of the direct product act smoothly and effectively in the j 'th coordinate, ball and trivially outside it. ■

4 The Epstein-Thurston obstruction to effective solvable actions

In this section G can be either real or complex. In the complex case an n -action means a holomorphic action on a complex n -dimensional manifold.

D.B.A. Epstein and W.P. Thurston [11, Theorem 1.1] discovered a fundamental necessary condition for effective local actions of solvable Lie groups:⁶

⁶The authors point out that their proof, stated for the real field, is valid in any category having a “good” dimension theory. It probably works for algebraic actions over arbitrary fields.

Theorem 4.1 *Assume G is solvable and has an effective local n -action. Then $n \geq \ell(G) - 1$, and $n \geq \ell(G)$ if G is nilpotent.*

The same conclusions hold for solvable Lie algebras actions. It turns out that in the borderline dimensions there are further restrictions on the structure of G and its orbits:

Theorem 4.2 *Assume G is solvable with derived length l . Let (G, M^n, α) be a nondegenerate local action, with $n = l$ if G is nilpotent and $n = l - 1$ otherwise.*

(i) *The union W of the open orbits is dense.*

(ii) *Suppose $G^{(l-1)}$ lies in the center C of G . Then $\dim(G^{(l-1)}) = \dim(C) = 1$, and $G^{(l-1)} = C$.*

Proof (i) $G^{(l-1)}$ acts trivially in each orbit of dimension $< n$ by the Epstein-Thurston theorem. As the action is effective, there is an orbit U in which $G^{(l-1)}$ acts nontrivially. The Epstein-Thurston theorem implies U is n -dimensional. This shows that W is nonempty. Each orbit $M \setminus W$ has dimension $< n$, therefore $G^{(l-1)}$ acts trivially in $M \setminus W$. Nondegeneracy implies $M \setminus W$ contains no open set, hence W is dense.

(ii) Fix a 1-dimensional subspace $Z \subset C$. In view of (i) we assume α is transitive and the orbits of Z^α are the fibres of a trivial fibration of $\pi: M^n \rightarrow V^{n-1}$. Let $(G/Z, V^{n-1}, \beta)$ be the action related equivariantly to α by π . The Epstein-Thurston theorem implies $\beta|_{G^{(l-1)}}$ is trivial, hence α -orbits of $G^{(l-1)} \times Z$ are the 1-dimensional orbits of Z^α . Let $K \subset G^{(l-1)} \times Z$ be the stabilizer of some point of M^n under the action of $\alpha|(G^{(l-1)} \times Z)$. Centrality of $G^{(l-1)} \times Z$ and transitivity of α imply K^α stabilizes every point of M^n , and is therefore trivial because α is effective. Consequently $\dim(G^{(l-1)} \times Z) = 1$, which implies (ii) because $G^{(l-1)}$ is nontrivial by the Epstein-Thurston Theorem. ■

Examination of the proof yields:

Corollary 4.3 *Assume G, l and n satisfy the hypothesis of Theorem 4.2, the center C of G contains $G^{(l-1)}$, and $\dim C > 1$. Then the kernel of any analytic n -action of G contains a 1-dimensional central subgroup.* ■

Example 4.4

For all $n \geq 2$ the group $N(n, \mathbb{F}) := ST(n, \mathbb{F})' \times \mathbb{F}$ has effective smooth actions on every n -manifold (Theorem 3.3). The actions constructed in the proof are highly degenerate, and in fact:

- *Every n -action of $N(n, \mathbb{F})$ is degenerate.*

This follows from Theorem 4.2(ii): $N(n, \mathbb{F})$ is nilpotent and with derived length n , and its center is 2-dimensional (over \mathbb{F}) and contains the 1-dimensional subgroup N_n^{n-1} .

5 Semisimple actions

Let (G, M, α) be an analytic action of a semisimple group. If the linearization $d\alpha_p$ at $p \in \text{Fix}(G)$ is trivial then α is trivial, because in a neighborhood of p , α is analytically equivalent to $d\alpha_p$ (A. Kušnirenko [35], V. Guillemin & S. Sternberg [16], R. Hermann [22]).

Cairns & Ghys [7] constructed effective C^∞ actions of $SL(2, \mathbb{R})$ on \mathbb{R}^3 and $SL(3, \mathbb{R})$ on \mathbb{R}^8 with fixed points at the origin, at which they are not topologically locally conjugate to analytic actions. Nevertheless the same conclusion holds for C^1 actions by a striking result, W. P. Thurston's "Generalized Reeb Stability Theorem," [56]:

Theorem 5.1 (THURSTON) *If α is a nontrivial local C^1 action of a semisimple Lie group, at every fixed point p the linearized action $d\alpha_p$ is nontrivial.*

Proof While Thurston states his theorem for global Lie group actions, the proof is entirely local. ■

Other results on semisimple actions are given in the papers cited above, and in T. Asoh [3], C. Schneider [50], D. Stowe [53], Uchida [59, 60], Uchida & Mukoyama [61].

Example 4.4 showed that all n -actions of $ST(n, \mathbb{R})' \times \mathbb{R}$ are degenerate. This phenomenon cannot occur for effective C^1 actions by semisimple groups:

Theorem 5.2 *Let G be a semisimple Lie group and (G, M^n, α) an effective C^1 local action. Then α is nondegenerate, as is the induced action in ∂M^n .*

Proof It suffices to consider a C^1 local group action (G, M^n, α) . For every invariant set $L \subset M$ let (G, L, α_L) be the action induced by α . Fix a nonempty open set $U \subset M^n$ and let $K \subset G$ denote the kernel of α_U . Every point $p \in U$ is a fixed point of $\alpha|_K$ at which the linearized action $d\alpha_p|_{\mathfrak{k}}$ is trivial. As K is normal in G and therefore semisimple, Thurston's theorem applied to $(K, M^n, \alpha|_K)$ shows that $K \subset \ker(\alpha)$. Therefore K is the trivial subgroup because α is effective, proving that α is nondegenerate.

Assume *per contra* that $\alpha_{\partial M}$ is degenerate. The preceding paragraph shows that there is a nontrivial proper normal subgroup $H \subset G$ such that H^α acts trivially on ∂M . Let (H, M^n, γ) be the action induced by α . At every $p \in \partial M^n$ there is an analytic coordinate chart centered at p taking a neighborhood of p onto an open subset of the origin in the closed half-space of \mathbb{R}^n defined by $x_n \geq 0$. In these coordinates $d\gamma_p$ represents H in the abelian subgroup comprising the matrices $A \in GL(\mathbb{R}^n)$ having the block form $\begin{bmatrix} I_{n-1} & b \\ 0 & 1 \end{bmatrix}$. Semisimplicity of H implies $d\gamma_p$ is trivial. Therefore γ is trivial by Thurston's theorem, contradicting effectiveness of α . ■

Here is another application of Thurston's result:

Theorem 5.3 *If $\partial M^n \neq \emptyset$, every C^1 local action $(SL_\circ(n+1, \mathbb{R}), M^n, \alpha)$ is trivial.*

Proof The Epstein-Thurston Theorem 4.1 implies the subgroup $ST_\circ(n+1, \mathbb{R})$ does not have effective local actions on $(n-1)$ -manifolds. Therefore $\alpha_{\partial M^n}$ is degenerate, so α is trivial by Theorem 5.2. ■

Example 5.4

Theorem 5.3 shows that $\widetilde{SL}_\circ(2, \mathbb{R})$ does not have effective C^1 local actions on the compact interval $[0, 1]$. On the other hand:

- $\widetilde{SL}_o(2, \mathbb{R})$ has nondegenerate continuous actions on $[0, 1]$.

To construct such an action, identify the open unit interval $]0, 1[$ with a universal covering space of \mathbf{S}^1 , lift the natural action $(SL_o(2, \mathbb{R}), \mathbf{S}^1, \alpha)$ to an action $(\widetilde{SL}_o(2, \mathbb{R}),]0, 1[, \beta)$, and extend β to an action $(\widetilde{SL}_o(2, \mathbb{R}),]0, 1[, \delta)$.

By putting δ on each radius of the compact n -disk \mathbf{D}^n , for every n we get a nondegenerate action of $\widetilde{SL}_o(2, \mathbb{R})$ on \mathbf{D}^n that is trivial on $\partial\mathbf{D}^n$. This leads to:

- $\widetilde{SL}_o(2, \mathbb{R})$ acts nondegenerately on all CW-complexes.⁷

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⁷Professor Palais said this construction is a “cheat”.

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